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Bose–Einstein condensation of a two-parameter deformed quantum group boson gas

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Abstract
We investigate the low temperature behaviour of a two-parameter generalized bosonic quantum gas with $SU_{q_1/q_2}(2)$-symmetry, where q_1 and q_2 are real independent deformation parameters. We calculate, in the thermodynamical limit, several statistical and thermodynamical functions of the system through an $SU_{q_1/q_2}(2)$-invariant bosonic Hamiltonian. In the low and high temperature limits, the specific heat of the system is obtained in terms of some functions of the deformation parameters q_1 and q_2. At the critical temperature being higher than that of the free boson gas, the specific heat of the two-parameter generalized boson gas exhibits a λ-point transition behaviour. We also discuss the conditions under which the Bose–Einstein condensation would occur in the present two-parameter generalized boson model. However, the free boson gas results can be recovered in the limit $q_1 = q_2 = 1$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum groups and their associated algebras, which are specific deformations of classical Lie groups and Lie algebras with some deformation parameter q (real or complex), have recently attracted a great deal of interest from both mathematicians and physicists. Originally, they emerged in studies of the quantum inverse scattering problem method and in the solution of the quantum Yang–Baxter equation [1]. These quantum groups and quantum algebras have been applied to several research areas of physics including two-dimensional conformal field theories [2], exactly solvable statistical models [3], noncommutative geometry [4] and the nuclear quantum many-body problem [5].

On the other hand, another direction of applications has focused on the field of statistical mechanics. For instance, possible connections between quantum groups and generalized statistical machanics have been studied [6]. We should also add that some earliear studies have been conducted to obtain a physical interpretation of the deformation parameter q by considering a canonical ensemble of q-oscillators [7], which are q-deformations of the bosonic harmonic oscillator algebra. Meanwhile, statistical and thermodynamical properties of a gas of q-deformed bosons as well as fermions have been extensively investigated in the literature [8]. Furthermore, some two-parameter generalizations of thermodynamical characteristics of a gas of (p, q)-deformed bosons without interaction have also been realized [9]. The high and low temperature thermodynamics of the one-parameter deformed boson gas having the symmetry of the quantum group $SU_q(2)$ have recently been studied by Ubriaco [10–12]. In these studies, the interactions among the bosonic particles of the system are fixed by the quantum group $SU_q(2)$. In fact, such interactions result from the quantum group invariance of the Hamiltonian of the system. The results of such studies have recently been generalized to the two-parameter deformed quantum group boson and fermion gas models [13–15]. However, a different bosonic generalization has been carried out by considering the two-parameter quantum group $GL_{p,q}(2)$-invariant boson gas defined under the condition $p = q^*, (p, q) \in C$ by Jellal [16].

In particular, the high temperature behaviour of a fermionic gas whose particle algebra is covariant under the quantum group $SU_{p/q}(2)$ has a crucial importance [14]. It is shown that this two-parameter $SU_{p/q}(2)$-fermion model in two spatial dimensions exhibits remarkably an anyonic type of behaviour at some critical values of the deformation parameters q and p. However, it is impossible to obtain a similar behaviour neither in the one-parameter deformed $SU_q(2)$-fermion gas model [10] nor in the free fermion gas model. Such interesting results have thus motivated us to study thermodynamical and statistical aspects of a bosonic version of the $SU_{p/q}(2)$-fermion model.

In this paper, we aim to study the thermodynamical properties of a bosonic gas having the symmetry of the quantum group $SU_{q_1/q_2}(2)$. The results obtained in this way will serve as a two-parameter generalization related to the low temperature thermodynamics of earlier quantum group boson gas studies [10–12]. We start with a bosonic Hamiltonian invariant under the quantum group $SU_{q_1/q_2}(2)$. We construct this Hamiltonian from the operators generating a two-parameter deformed $SU_{q_1/q_2}(2)$-invariant boson algebra. Obviously, this algebra becomes the ordinary boson algebra in the limit $q_1 = q_2 = 1$. We then investigate the low temperature (high density) behaviour of such a two-parameter deformed boson model with $SU_{q_1/q_2}(2)$-symmetry in the thermodynamical limit. Particularly, by considering the role of deformation parameters q_1 and q_2, we discuss the conditions under which the Bose–Einstein condensation would occur.

The paper is organized as follows. In section 2, we review some basic definitions and properties concerning the quantum group $SU_q(2)$-bosons, and present its two-parameter generalization $SU_{q_1/q_2}(2)$-bosons. In section 3, we introduce our model described by an $SU_{q_1/q_2}(2)$-invariant bosonic Hamiltonian. This leads to the discussion of thermodynamics of the model obtained via the grand partition function given in section 4. In particular, we find, in the thermodynamical limit, the average number of particles, the critical temperature T_c and the internal energy U of the $SU_{q_1/q_2}(2)$-boson gas. In the low and high temperature limits, we then calculate the specific heat of the system in terms of some functions of the deformation parameters q_1 and q_2. In the last section, we discuss the phenomenon of Bose–Einstein condensation in the present two-parameter $SU_{q_1/q_2}(2)$-boson model and give our conclusions.

2. Quantum group $SU_{q_1/q_2}(2)$-bosons

In this section, we recall the general properties of the $SU_q(N)$-bosons and present its two-parameter generalization. The usual boson oscillators satisfy the following commutation relations:

$$\phi_i\phi_j^* - \phi_j^*\phi_i = \delta_{ij}, \qquad \phi_i\phi_j - \phi_j\phi_i = 0, \qquad \phi_i^*\phi_i = N_i, \quad i, j = 1, 2, \ldots, N, \tag{1}$$

where ϕ_i and ϕ_i^* are the bosonic annihilation and creation operators, respectively, and N_i is the boson number operator. These oscillators are invariant under the action of the undeformed group $SU(N)$. The quantum group analogues of these relations are defined by the following commutation relations [17]:

$$\Phi_j\Phi_i^* = \delta_{ij} + qR_{kijl}\Phi_l^*\Phi_k, \tag{2}$$

$$\Phi_l\Phi_k = q^{-1}R_{jikl}\Phi_j\Phi_i, \qquad i, j = 1, 2, \ldots, N, \tag{3}$$

where the $N^2 \times N^2$ matrix R_{jikl} [4] is

$$R_{jikl} = \delta_{jk}\delta_{il}(1 + (q-1)\delta_{ij}) + (q - q^{-1})\delta_{ik}\delta_{jl}\theta(j-i), \tag{4}$$

and the function $\theta(j-i) = 1$ for $j > i$ and zero otherwise. Under the linear transformation

$$\Phi_i' = \sum_{j=1}^{N} T_{ij}\Phi_j, \tag{5}$$

where the matrix $T \in SU_q(N)$, the relations given in equations (2) and (3) are invariant. The $SU_q(N)$ transformation matrix T and the R matrix satisfy the following relations [18]:

$$RT_1T_2 = T_2T_1R, \tag{6}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{7}$$

where $T_1 = T \otimes 1$, $T_2 = 1 \otimes T \in V \otimes V$ and $(R_{23})_{ijk,i'j'k'} = \delta_{ii'}R_{jk,j'k'} \in V \otimes V \otimes V$.

It can be obtained from equation (6) through a unitary quantum group matrix [19]

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

that

$$\begin{array}{lll} ab = qba, & ac = qca & cd = qdc, \\ bd = qdb & bc = cb, & ad - da = (q - q^{-1})bc \\ \mathrm{Det}_q(T) = ad - qbc = 1, & & \end{array} \tag{8}$$

with the unitary conditions $a^* = d$, $b^* = -qc$, $q \in R$. In paticular, for $N = 2$, the $SU_q(2)$-invariant algebra generated by the quantum group bosons Φ_i, $i = 1, 2$, is given by the following relations [17]:

$$\begin{aligned} &\Phi_1\Phi_1^* - q^2\Phi_1^*\Phi_1 = 1, \\ &\Phi_2\Phi_2^* - q^2\Phi_2^*\Phi_2 = 1 + (q^2 - 1)\Phi_1^*\Phi_1, \\ &\Phi_1\Phi_2 = q\Phi_2\Phi_1, \\ &\Phi_1\Phi_2^* = q\Phi_2^*\Phi_1, \end{aligned} \tag{9}$$

in which they become the usual boson algebra in the limit $q = 1$. However, we exploit a different quantum group, $SU_r(2)$-bosons, where $r = q_1/q_2$. Although two-parameter extensions of such investigations have been studied earlier [20], the two-parameter deformed $SU_{q_1/q_2}(2)$-invariant bosonic oscillator algebra was first introduced in [21] during a realization of the most general quantum group invariant oscillator algebra.

The two-parameter generalized bosonic quantum gas with $SU_{q_1/q_2}(2)$-symmetry is generated by the quantum group invariant bosonic Φ_i oscillators and defined by the following deformed commutation relations [21]:

$$\begin{aligned}
&\Phi_1\Phi_1^* - q_1^2\Phi_1^*\Phi_1 = q_2^{2N},\\
&\Phi_2\Phi_2^* - q_1^2\Phi_2^*\Phi_2 = q_2^{2N} + \left(q_1^2 - q_2^2\right)\Phi_1^*\Phi_1,\\
&\Phi_1\Phi_2 = q_1q_2^{-1}\Phi_2\Phi_1,\\
&\Phi_1^*\Phi_2^* = q_2q_1^{-1}\Phi_2^*\Phi_1^*,\\
&\Phi_1\Phi_2^* = q_1q_2\Phi_2^*\Phi_1,
\end{aligned} \tag{10}$$

where N is the total boson number operator and $q_{1,}$ q_2 are the real independent deformation parameters. Hereafter, we will consider $0 < q_1 < \infty$ and $0 < q_2 < \infty$. Moreover, the total deformed number operator for these two-parameter deformed oscillators is

$$\Phi_1^*\Phi_1 + \Phi_2^*\Phi_2 = [N_1 + N_2] = [N], \tag{11}$$

whose spectrum is defined by the following Fibonacci basic number $[n]$:

$$[n] = \frac{q_2^{2n} - q_1^{2n}}{q_2^2 - q_1^2}, \tag{12}$$

which is a generalization of the usual q-numbers. Since all thermodynamical and statistical functions in this study are obtained in terms of this generalized q-number, the Fibonacci basic number $[n]$ will be of crucial importance for the present two-parameter boson gas model.

With the above theoretical motivation in mind, one can check that the two-parameter deformed bosonic algebra in equations (10) and (11) shows $SU_{q_1/q_2}(2)$-symmetry. Our deformed bosonic algebra is invariant under the following transformation:

$$\begin{pmatrix}\Phi_1'\\ \Phi_2'\end{pmatrix} = T\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}a & -q_1q_2^{-1}b^*\\ b & a^*\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \tag{13}$$

such that T is the transformation matrix and $T \in SU_{q_1/q_2}(2)$. The elements of matrix T satisfy the following equations:

$$\begin{aligned}
&ab = q_1q_2^{-1}ba, && ab^* = q_1q_2^{-1}b^*a,\\
&bb^* = b^*b, && aa^* + q_1^2q_2^{-2}b^*b = 1,\\
&a^*a + bb^* = 1.
\end{aligned} \tag{14}$$

If we can rewrite all relations in equations (10) and (11) for the transformed ones, one can readily see that our system remains unchanged. We note that the matrix elements of T are assumed to commute with Φ_1, Φ_2, Φ_1^*, Φ_2^*.

Before closing this section, we should also emphasize some important limiting cases of the $SU_{q_1/q_2}(2)$-invariant bosonic oscillator algebra in equations (10) and (11). The one-parameter deformed $SU_q(2)$-invariant bosonic oscillator algebra can be obtained in the limit $q_2 = 1$ as defined by equation (9). In the limit $q_1 = q_2 = 1$, one can recover the usual bosonic algebra in equation (1).

3. $SU_{q_1/q_2}(2)$-boson model

In this section, our aim is to find a representation of Φ_i oscillators in terms of the usual bosonic oscillators ϕ_i. First, we consider the following Hamiltonian in terms of $SU_{q_1/q_2}(2)$-generators for two different kinds of bosonic particle families with the same energy,

$$H_B = \sum_k \varepsilon_k(M_{1,k} + M_{2,k}), \tag{15}$$

where the deformed boson number operators $M_{1,k}$ and $M_{2,k}$ are defined as

$$M_{1,k} = \Phi^*_{1,k}\Phi_{1,k}, \qquad M_{2,k} = \Phi^*_{2,k}\Phi_{2,k}, \tag{16}$$

ε_k is the spectrum of energy, $k = 0, 1, 2, \ldots$, and $[\Phi^*_{i,k}, \Phi_{j,k'}] = 0$, for $k \neq k'$. The operators M_1 and M_2 satisfy the following relations for a given k:

$$M_2\Phi_1 - q_1^{-2}\Phi_1 M_2 = 0, \qquad M_1\Phi_2 - q_2^{-2}\Phi_2 M_1 = 0. \tag{17}$$

The normalized states of the above Hamiltonian can be built by applying the operators Φ^* on the vacuum state $|0, 0\rangle$ for a given k as

$$|m_1, m_2\rangle = 1/\sqrt{[m_1]![m_2]!}\,\Phi_1^{*^{m_1}}\Phi_2^{*^{m_2}}|0, 0\rangle, \tag{18}$$

where the Fibonacci basic number $[m]$ is defined in equation (12). In order to express a new representation for Φ_i oscillators in terms of the usual bosonic oscillators $\phi_{i,k}$ and $\phi^*_{i,k}$ satisfying equation (1), we propose the following representations for a given k:

$$\Phi_1 = (\phi_1^*)^{-1}[N_1]q_2^{N_2}, \qquad \Phi_1^* = \phi_1^* q_2^{N_2}, \tag{19}$$

$$\Phi_2 = (\phi_2^*)^{-1}[N_2]q_1^{N_1}, \qquad \Phi_2^* = \phi_2^* q_1^{N_1}. \tag{20}$$

By means of this representation, we are able to rewrite the Hamiltonian in equation (15) as

$$H_B = \sum_k \varepsilon_k[N_1 + N_2], \tag{21}$$

where $N_{i,k} = \phi^*_{i,k}\phi_{i,k}$ and the spectrum of the bracket $[N_1 + N_2]$ is given by equation (12). It is important to note that when we compare this new Hamiltonian with the original Hamiltonian in equation (15), this representation brings about an interacting Hamiltonian for the system of two different kinds of bosonic particle families. This results from the two-parameter quantum group symmetry of the system. Also, such an interaction is fixed by the deformation parameters q_1 and q_2. The non-interacting system generated by free bosonic particles can obviously be obtained in the limit $q_1 = q_2 = 1$. The low temperature thermodynamics of a gas of such two-parameter deformed quantum group invariant bosonic oscillators will be discussed in the next section.

On the other hand, we would like to generalize the representations given in equations (19) and (20) for arbitrary N case via the following transformations:

$$\begin{aligned}
\Phi_1 &= (\phi_1^*)^{-1}[N_1]q_2^{\sum_{l=2}^{N} N_l},\\
\Phi_2 &= (\phi_2^*)^{-1}q_1^{N_1}[N_2]q_2^{\sum_{l=3}^{N} N_l},\\
\Phi_3 &= (\phi_3^*)^{-1}q_1^{N_1+N_2}[N_3]q_2^{\sum_{l=4}^{N} N_l},\\
&\ \vdots\\
\Phi_m &= (\phi_m^*)^{-1}q_1^{\sum_{l=1}^{N-1} N_l}[N_m],
\end{aligned} \tag{22}$$

and for the adjoint equations

$$\begin{aligned}
\Phi_1^* &= (\phi_1^*)q_2^{\sum_{l=2}^{N} N_l},\\
\Phi_2^* &= q_1^{N_1}(\phi_2^*)q_2^{\sum_{l=3}^{N} N_l},\\
\Phi_3^* &= q_1^{N_1+N_2}(\phi_3^*)q_2^{\sum_{l=4}^{N} N_l},\\
&\ \vdots\\
\Phi_m^* &= q_1^{\sum_{l=1}^{N-1} N_l}(\phi_m^*).
\end{aligned} \tag{23}$$

4. Low temperature thermodynamics of $SU_{q_1/q_2}(2)$-boson gas

In this section, we investigate the low temperature (high density) behaviour of the $SU_{q_1/q_2}(2)$-boson gas described by the Hamiltonian in equation (21). This Hamiltonian can also be written as

$$H_B = \sum_k \frac{\varepsilon_k}{(q_1^2 - q_2^2)} \sum_{l=1}^{\infty} \frac{2^l (N_1 + N_2)^l}{l!} (\ln^l q_1 - \ln^l q_2). \tag{24}$$

The grand partition function Z_B of the system is

$$Z_B = \mathrm{Tr}\, \exp\left[-\beta \sum_k \varepsilon_k (\Phi_{1,k}^* \Phi_{1,k} + \Phi_{2,k}^* \Phi_{2,k}) \right] \mathrm{e}^{\beta\mu(N_{1,k}+N_{2,k})}, \tag{25}$$

where the trace is taken over the states in equation (18). From equations (19)–(21), this grand partition function becomes

$$Z_B = \prod_k \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathrm{e}^{-\beta\varepsilon_k [n_1+n_2]} \, \mathrm{e}^{\beta\mu(n_1+n_2)}, \tag{26}$$

which can be rewritten as

$$Z_B = \prod_k \sum_{n=0}^{\infty} (n+1)\, \mathrm{e}^{-\beta\varepsilon_k [n]} z^n, \tag{27}$$

where $z = \mathrm{e}^{\beta\mu}$ is the fugacity. Since we are studying the low temperature (high density) behaviour of the $SU_{q_1/q_2}(2)$-boson gas, we write the above grand partition function in the thermodynamical limit as

$$\ln Z_B = \ln\left(1 + \sum_{n=1}^{\infty} (n+1) z^n\right) + \frac{4\pi V}{h^3} \int_0^{\infty} p^2 \mathrm{d}p \ln\left(1 + \sum_{n=1}^{\infty} (n+1)\, \mathrm{e}^{-\beta[n]\varepsilon} z^n\right). \tag{28}$$

We note that the $\vec{p} = \vec{0}$ case plays a special role in the ideal Bose gas [22]. Since $\ln Z_B$ diverges in the $\vec{p} = \vec{0}$ term as $z \to 1$, we separately account for the term $\vec{p} = \vec{0}$ as the first term in the sum of equation (28). Although the quantum algebraic structure of the $SU_{q_1/q_2}(2)$-invariant boson model defined by equations (10)–(12) is symmetric between q_1 and q_2, the only limitation for the convergence of the series in the integrand of equation (28) comes conventionally from the definition of the quantum group $SU_r(2)$ with $0 < r \leqslant 1$. Since $r = q_1/q_2$ for our model, we should have the condition $q_2 > q_1$ for the rest of the calculations of this study.

One can calculate the average number of particles $\langle N \rangle$ by

$$\langle N \rangle = \beta^{-1} (\partial \ln Z_B / \partial\mu)_{T,V}, \tag{29}$$

which leads to

$$\langle N \rangle = \langle N_0 \rangle + \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{z^n}{[n]^{3/2}}, \tag{30}$$

where the thermal wavelength is $\lambda = \sqrt{(2\pi\hbar^2/mkT)}$. Equation (30) can also be written as

$$\lambda^3 \frac{\langle N_0 \rangle}{V} = \frac{\lambda^3}{v} - \tilde{g}_{3/2}(z, q_1, q_2), \tag{31}$$

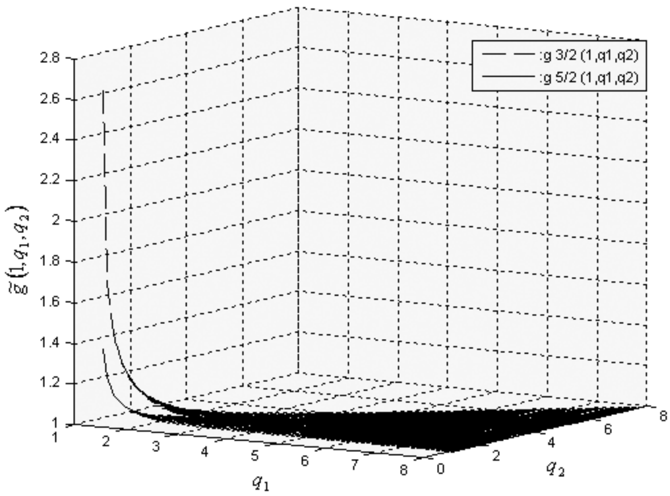

Figure 1. The (q_1, q_2)-deformed functions $\tilde{g}_{3/2}(1, q_1, q_2)$ and $\tilde{g}_{5/2}(1, q_1, q_2)$ as a function of the model parameters q_1 and q_2 for $1 \leqslant q_1 \leqslant 8$, $1 \leqslant q_2 \leqslant 8$.

where $\nu = V/\langle N\rangle$ and the (q_1, q_2)-deformed function $\tilde{g}_{3/2}(z, q_1, q_2)$ is defined by

$$\tilde{g}_{3/2}(z, q_1, q_2) = \sum_{n=1}^{\infty} \frac{z^n}{[n]^{3/2}}, \tag{32}$$

with the Fibonacci basic number $[n]$. In the limit $q_1 = q_2 = z = 1$, we find the function $\tilde{g}_{3/2}(1, 1, 1) = \zeta(3/2)$, which is the Riemann zeta function. Our model will exhibit the Bose–Einstein condensation when the following condition is satisfied:

$$\frac{\lambda^3}{\nu} \geqslant \tilde{g}_{3/2}(1, q_1, q_2). \tag{33}$$

This means that a finite fraction of the particles occupies the level with $\vec{p} = \vec{0}$. The value of the (q_1, q_2)-deformed function $\tilde{g}_{3/2}(1, q_1, q_2)$ depends on the deformation parameters q_1 and q_2. Therefore, these parameters are responsible for the low temperature behaviour of the present two-parameter boson gas model. Figure 1 shows a plot of the (q_1, q_2)-deformed function $\tilde{g}_{3/2}(1, q_1, q_2)$ as a function of the model parameters q_1 and q_2.

The critical temperature $T_c(q_1, q_2)$ for our model can be found from equation (33) as

$$T_c(q_1, q_2) = \frac{2\pi\hbar^2/mk}{\left[\nu \tilde{g}_{3/2}(1, q_1, q_2)\right]^{2/3}}. \tag{34}$$

Thus, the (q_1, q_2)-deformed function $\tilde{g}_{3/2}(1, q_1, q_2) < \tilde{g}_{3/2}(1, 1, 1) = \zeta(3/2) = 2.61$, which means that the critical temperature for our model is much larger than the critical temperature $T_c(1, 1)$ for a free boson gas. Moreover, we compare these critical temperatures with those obtained from the one-parameter deformed $SU_q(2)$-boson model in [11] as follows:

$$T_c(q_1, q_2) > T_c(q) > T_c(1, 1). \tag{35}$$

Obviously, one can find a relation between the critical temperatures of the present two-parameter boson gas model and the free boson gas:

$$\frac{T_c(q_1, q_2)}{T_c(1, 1)} = \left(\frac{2.61}{\tilde{g}_{3/2}(1, q_1, q_2)}\right)^{2/3}. \tag{36}$$

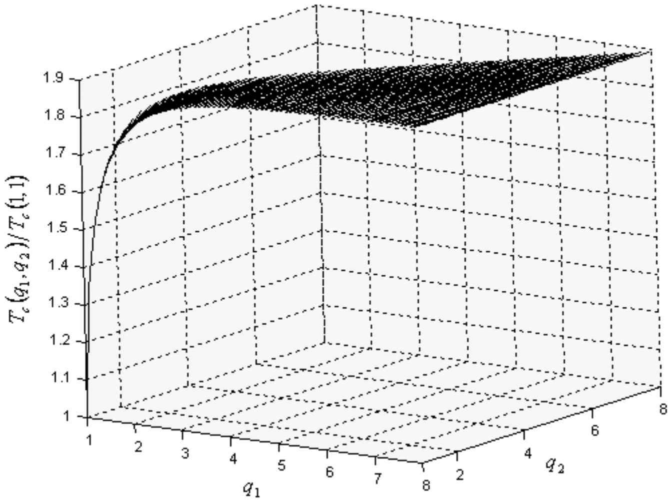

Figure 2. The ratio $T_c(q_1, q_2)/T_c(1, 1)$ of the (q_1, q_2)-deformed critical temperature $T_c(q_1, q_2)$ and the undeformed $T_c(1, 1)$ as a function of the deformation parameters q_1, q_2.

In figure 2, we show the plot of equation (36) as a function of the deformation parameters q_1 and q_2.

The internal energy U of the system can be calculated from

$$U = -\frac{\partial \ln Z_B}{\partial \beta} + \mu\langle N\rangle, \tag{37}$$

which leads to

$$\frac{U}{\langle N\rangle} = \frac{3}{2}\frac{\nu kT}{\lambda^3}\tilde{g}_{5/2}(z, q_1, q_2), \tag{38}$$

where the (q_1, q_2)-deformed function $\tilde{g}_{5/2}(z, q_1, q_2)$ is defined as

$$\tilde{g}_{5/2}(z, q_1, q_2) = \sum_{n=1}^{\infty} \frac{z^n}{[n]^{5/2}}. \tag{39}$$

When we take the limit $q_1 = q_2 = z = 1$, the function $\tilde{g}_{5/2}(1, 1, 1) = \zeta(5/2) = 1.34$. Figure 1 also shows a graph of the (q_1, q_2)-deformed function $\tilde{g}_{5/2}(1, q_1, q_2)$ as a function of the model parameters q_1 and q_2. With the above results in mind, the specific heat of the $SU_{q_1/q_2}(2)$-boson gas for temperatures $T < T_c(q_1, q_2)$ can be obtained from $C_V = (\partial U/\partial T)_V$. For low temperatures, we have the chemical potential $\mu = 0$. Consequently, the specific heat of our model for temperatures $T < T_c(q_1, q_2)$ is

$$\frac{C_V}{k\langle N\rangle} = \frac{15}{4}\frac{\tilde{g}_{5/2}(1, q_1, q_2)}{\tilde{g}_{3/2}(1, q_1, q_2)}\left(\frac{T}{T_c(q_1, q_2)}\right)^{3/2}. \tag{40}$$

We now wish to summarize some of the results of [13] in a slightly different form in order to find the specific heat of our model in the high temperature limit, i.e., the limit $T > T_c(q_1, q_2)$. In this limit, we have

$$\ln Z_B = \frac{V}{\lambda^3}(2z + 4z^2\xi(q_1, q_2) + \cdots), \tag{41}$$

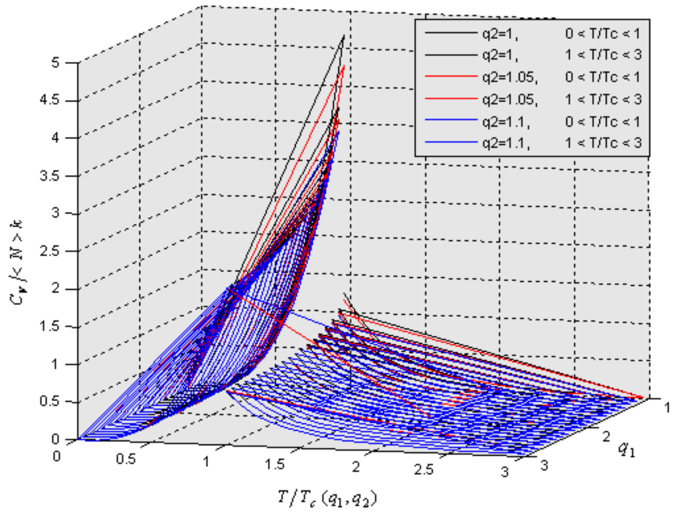

Figure 3. The specific heat $C_v/\langle N\rangle k$ as a function of $T/T_c(q_1, q_2)$ for various values of the deformation parameters q_2 and $1 \leqslant q_1 \leqslant 3$.

where the function $\xi(q_1, q_2)$ is

$$\xi(q_1, q_2) = \frac{1}{4}\left[\frac{3}{(q_1^2 + q_2^2)^{3/2}} - \frac{1}{\sqrt{2}}\right]. \tag{42}$$

From equation (41), we find the fugacity for high temperatures as

$$z \approx \frac{\langle N\rangle}{2V}\lambda^3\left(1 - \frac{2\lambda^3\xi(q_1, q_2)\langle N\rangle}{V} + \cdots\right), \tag{43}$$

and the internal energy in this case is

$$U = \frac{3}{2}\frac{\langle N\rangle}{\beta}\left[1 - \frac{\lambda^3\xi(q_1, q_2)\langle N\rangle}{V} + \cdots\right]. \tag{44}$$

Therefore, the specific heat for the limit $T > T_c(q_1, q_2)$ is

$$\frac{C_V}{k\langle N\rangle} = \frac{3}{2}\left(1 + \frac{\lambda^3\xi(q_1, q_2)\langle N\rangle}{2V} + \cdots\right). \tag{45}$$

Using equation (34), we can rewrite the above equation as

$$\frac{C_V}{k\langle N\rangle} = \frac{3}{2}\left[1 + \frac{1}{2}\xi(q_1, q_2)\tilde{g}_{3/2}(1, q_1, q_2)\left(\frac{T_c(q_1, q_2)}{T}\right)^{3/2} + \cdots\right]. \tag{46}$$

From equations (40) and (46), we deduce the gap in the specific heat in the limit $T = T_c(q_1, q_2)$ as follows:

$$\frac{\Delta C_v}{k\langle N\rangle} \approx \left\{\frac{15}{4}\frac{\tilde{g}_{5/2}(1, q_1, q_2)}{\tilde{g}_{3/2}(1, q_1, q_2)} - \frac{3}{2}\left[1 + \frac{1}{2}\xi(q_1, q_2)\tilde{g}_{3/2}(1, q_1, q_2)\right]\right\}. \tag{47}$$

In figures 3 and 4, we show the plots of the specific heat $C_V/k\langle N\rangle$ as a function of $T/T_c(q_1, q_2)$ for several values of the deformation parameters q_1 and q_2.

The pressure for low temperatures can be obtained from $P = \beta^{-1}(\partial \ln Z_B/\partial V)_{T,\mu}$ as

$$P(1, q_1, q_2) = kT\lambda^{-3}\tilde{g}_{5/2}(1, q_1, q_2). \tag{48}$$

By considering the above results, the effect of two deformation parameters on the thermodynamics of the system will be discussed in the next section.

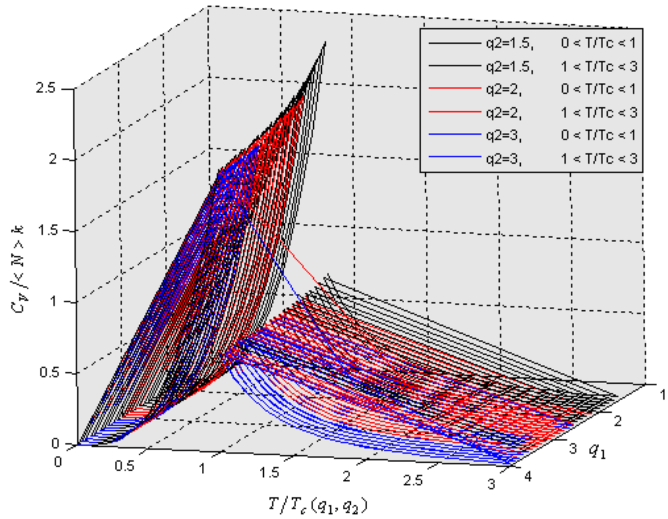

Figure 4. The specific heat $C_v/\langle N\rangle k$ as a function of $T/T_c(q_1, q_2)$ for various values of the deformation parameters q_2 and $1 \leqslant q_1 \leqslant 4$.

5. Discussion and conclusion

In this paper, we studied the thermostatistical consequences of introducing two-parameter quantum group symmetry to a gas of two different kinds of bosonic particles. In particular, we discussed the low temperature behaviour of a two-parameter deformed quantum group bosonic gas with $SU_{q_1/q_2}(2)$-symmetry. By means of an $SU_{q_1/q_2}(2)$-invariant bosonic Hamiltonian, we calculated several thermostatistical characteristics via the grand partition function of the system. We obtained such characteristics in terms of some functions of deformation parameters of the model. For instance, the average number of particles, the critical temperature, the internal energy and the pressure are derived for low temperatures. Subsequently, the specific heat of the system is obtained in the low and high temperature limits. We then focused on the effect of the deformation parameters q_1 and q_2 on these results. The (q_1, q_2)-deformed functions $\tilde{g}_{3/2}(1, q_1, q_2)$ and $\tilde{g}_{5/2}(1, q_1, q_2)$ in equations (32) and (39) are obtained as series expressed in terms of the Fibonacci basic numbers $[n]$. As shown in figure 1, these deformed functions become their respective Riemann zeta functions $\zeta(3/2)$ and $\zeta(5/2)$ in the limit $q_1 = q_2 = 1$. The values of these deformed functions decrease from their maximum values at $q_1 = q_2 = 1$ and become approximately constant for $q_1 = q_2 > 2$. However, we should emphasize that the values of these deformed functions and thus all other thermodynamical and statistical functions change more rapidly in the interval $1 \leqslant (q_1, q_2) \leqslant 2$.

On the other hand, as shown in figures 3 and 4, the specific heat of our model shows a discontinuity at the critical temperature. This is the reason why we called the Bose–Einstein condensation a second-order phase transition (no latent heat). Furthermore, the specific heat of the $SU_{q_1/q_2}(2)$-boson model has a λ-point transition behaviour which is not exhibited by the free boson gas. Such behaviour is one of the important characteristics of some physical phenomena such as superfluidity. An interesting point is that when the second deformation parameter q_2 increases, the discontinuity in the specific heat of the system decreases (figures 3 and 4) and it disappears in the limit $q_1 = q_2 = 1$, showing therefore a free bosonic gas behaviour.

Also, the gap in the specific heat of the $SU_{q_1/q_2}(2)$-boson gas at the condensation temperature decreases with the values of the deformation parameter q_2, and becomes approximately constant after the values $q_1 \geqslant 2.2$, $q_2 \geqslant 2.5$. Another interesting property of the present two-parameter deformed boson gas is that, for high temperatures, it behaves as a fermion gas [13] at the value of $\left(q_1^2 + q_2^2\right) \approx 4.16$ and it shows the Bose–Einstein condensation for low temperatures in the interval $q_2 > q_1 > 0$. Obviously, the results for the free boson gas can be found in the limit $q_1 = q_2 = 1$.

We now wish to discuss other effects of the deformation parameters q_1 and q_2 on the algebraic structure of a system of the $SU_{q_1/q_2}(2)$-invariant bosonic oscillators. These deformation parameters play important roles for the system under consideration. Not only do they constitute a quantum deformation of the classical symmetry group of the system, but also they describe an interaction between two bosonic particle families. We may interpret our model as containing two different kinds of bosonic oscillator families which interact with each other via the deformation parameters q_1 and q_2 fixed by the quantum group $SU_{q_1/q_2}(2)$-symmetry. But these two bosonic families do not interact among themselves. Therefore, in some sense, the entire behaviour of the system is characterized by the model parameters q_1 and q_2. When we take the limit $q_1 = q_2 = 1$, the non-interacting system with two different kinds of ordinary bosons can be recovered. In the limit $q_2 = 1$, these two bosonic particle families are also interacting via the deformation parameter q_1, but in this case, one of the bosonic oscillator families does not have the same physical properties as the other.

As a final remark, the low temperature behaviour of a fermionic version of the present two-parameter $SU_{q_1/q_2}(2)$-boson model could be another direction of this work. One other problem is how the analysis presented in this paper could be extended to a gas of relativistic bosonic particles with the same quantum group symmetry. Furthermore, it would be interesting to investigate the algebraic and statistical consequences of the present two-parameter $SU_{q_1/q_2}(2)$-boson model when the deformation parameter q_1/q_2 is a root of unity. We hope that these problems will be addressed in the near future.

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